1 Solving a system of linear algebraic equations

How do we solve a system of linear algebraic equations? A short answer is "by sequential elimination of unknowns." Arguably, the easiest way to show how to do it is by an example, so let me start with a simple example of three linear equations with three unknowns:

Example 1.1. Solve

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1, \\ 4x_1 + x_2 &= -2, \\ -2x_1 + 2x_2 + x_3 &= 7. \end{aligned} \tag{1.1}$$

By carefully expecting the equations I note that variable x_3 is missing in the second equation, and that by subtracting the third equation from the first one I can also eliminate x_3. Hence, I get

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1, \\ 4x_1 + x_2 &= -2, \\ 4x_1 - x_2 &= -6, \end{aligned} \tag{1.2}$$

where, as I said before, the third equation is obtained by taking the first equation in (1.1) and subtracting the third one from it. Now I clearly see that if I add the second and third equations in (1.2), I can get rid of x_2:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1, \\ 4x_1 + x_2 &= -2, \\ 8x_1 &= -8. \end{aligned} \tag{1.3}$$

My manipulations led to $8x_1 = -8$, or, by dividing both sides of this equality by 8, to

$$x_1 = -1.$$

I can now plug the found value of x_1 into the second equation in (1.3) to find[1] that

$$-4 + x_2 = -2 \Longrightarrow x_2 = 2,$$

where the symbol "$\Longrightarrow$" should be read as "implies." Finally, plugging the values $x_1 = -1$, $x_2 = 2$ into the first equation in (1.3) I get

$$-2 + 2 + x_3 = 1 \Longrightarrow x_3 = 1,$$

and hence my solution (I write my *vectors* as rows to save some space) is

$$[x_1 \ \ x_2 \ \ x_3] = [-1 \ \ 2 \ \ 1].$$

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[1]The word "algebra" comes from the arabic "al-jabr," which literally means "completion." This word was used in the title of the famous *The Compendious Book on Calculation by Completion and Balancing* by the persian mathematician Muḥammad ibn Mūsā al-Khwārizmī. The mathematical meaning of this term is that if we transfer one term of the equation to the other side, then this term should change its sign, which is exactly what I do in the lines after this footnote.

Very often this procedure of sequential elimination of variables is called *the Gaussian elimination.*

Is this sequence of steps that I undertook the only way to solve the original system? Of course, not. Note that in my solution I relied on my human choice of the sequence of steps, to make them as simple as possible arithmetically. This works great for small systems, which contain 3 or 4 equations and unknowns. For bigger systems an obvious simple initial step can lead to future complications, and hence it is advisable to come up with an *automatic* procedure, in the form of an algorithm, which would yield the sought solution. Here is how this can be done with the same system (1.1).

I will be doing exactly the same thing, namely, I will eliminate the variables one by one, but now, instead of looking what a *simpler* way is to eliminate the variables, I will do it in a fixed sequential order. Since I have a nonzero coefficient at x_1 in the first equation, I can eliminate x_1 from both the second and third equations. To do this I need first multiply the first equation by -2 and add it to the second one, and also simply add the first equation to the third one. I get

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1, \\ -x_2 - 2x_3 &= -4, \\ 3x_2 + x_3 &= 8. \end{aligned} \tag{1.4}$$

Now, since the first nonzero coefficient in the second equation is at x_2, I can eliminate x_2 in the third equation, by multiplying the second equation in (1.4) by 3 and adding it to the third one:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1, \\ -x_2 - 2x_3 &= -4, \\ -4x_3 &= -4. \end{aligned} \tag{1.5}$$

Now I can go backward: From the third equation in (1.5) I find $x_3 = 1$, this information and the second equation imply that $x_2 = 2$, and finally the first equation yields $x_1 = -1$. I arrived at the same solution.

Actually, it is quite boring to type all the variables all the time. The last procedure can be written entirely in terms of the coefficients and numbers in the system. For this I represent my original system as a table of numbers, whose structure should be obvious:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 4 & 1 & 0 & -2 \\ -2 & 2 & 1 & 7 \end{array}\right]. \tag{1.6}$$

Now the operations I performed can be written as (make sure that you understand what is done at each step)

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 4 & 1 & 0 & -2 \\ -2 & 2 & 1 & 7 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 3 & 2 & 8 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & -4 & -4 \end{array}\right], \tag{1.7}$$

from which the values of the unknowns can be found by *back-substitution.*

After Example 1.1 several questions naturally pop up.

- We found one solution, how can we be sure that there are no other solutions?
- Can this procedure of sequential elimination of variables break down?

- Can we solve in a similar way systems where the number of variables is not equal to the number of equations?

Let me start with the first question, the other questions will be answered in a due course. Note that system (1.3) necessarily has only one solution. Hence I can rephrase my first question as follows: "Do the systems (1.1), (1.2), (1.3) have the same solutions?" Note that they are different one from another by an *elementary operation* that consists of multiplying one equation by a non-zero constant and adding this equation to another one. Therefore, I can further rephrase my question, now in a much more general form: "Do two linear systems, such that one is obtained from the other by an elementary operation of multiplying one of the equations by a constant and adding it to another equation, have the same solutions?" The answer is "yes," but how to convince those that would not simply believe this answer?

The ultimate mathematical way of establishing that something is true is the concept of a *mathematical proof.* I will prove almost all the statements in this course, and my first proof will be the proof of the following *proposition* ("proposition" is a name in mathematics for a true statement or fact).

Proposition 1.2. *Consider two linear systems of n equations. Denote the first system as*

$$\begin{aligned} E_1 &= b_1, \\ E_2 &= b_2, \\ &\dots \\ E_m &= b_m, \end{aligned} \tag{1.8}$$

here E_i, $i = 1, \ldots, m$, are the abbreviation of the left hand sides of the equations in the system, the i-th equation in general has the form

$$E_i = a_{i1}x_1 + \ldots + a_{in}x_n = b_i,$$

that is, it is assumed that the number of equation, m, is not in general equal to the number of unknowns, n. The second linear system has as its j-th equation the following

$$E_j' = E_j + \alpha E_k = b_j + \alpha b_k,$$

i.e., it is obtained by multiplying the k-th equation in (1.8) *by a constant α and adding it to the j-th equations; all other equations of the second system coincide with the equations in* (1.8)*.*

Then these two systems have the same solutions.

Remark 1.3. I will say much more about the structure and techniques of proofs during this course. Here let me make two comments. The statements which mathematicians prove can have different names: theorems, lemmas, propositions, statements. "Theorem" usually refers to some important result. "Lemma" usually means an auxiliary fact, which will be used later is a proof. Not that important (or complicated) results can be called "propositions."

My second remark here is actually about what we need to prove. A usual structure of a mathematical statement that needs to be proved is of the form "if A then B," i.e., we usually need to show that if something is given and it is true (we do know this) then statement B is also true. If you carefully reread the proposition above you should see that it involves actually two goals of this form. In particular, it says that if $x_1, \ldots, x_n$ is a solution of (1.8) (let me abbreviate this statement as A) then it is also a solution of the second system (this is my statement B); the second goal is to show that if $(x_1, \ldots, x_n)$ is a solution of the second system then it is also a solution of the first one (more concisely, "if B then A"). Using the introduced notation $\Longrightarrow$, I can encrypt my statements as $A \Longrightarrow B$ and $B \Longrightarrow A$, which is further abbreviated as $A \Longleftrightarrow B$, in words, "A is equivalent to B" or "A if and only if B."

Now I am ready to prove my proposition.

Proof. ($A \Longrightarrow B$) Assume that $x_1, \ldots, x_m$ solve (1.8). This means that I am assuming that equations $E_i = b_i$ for all i turn into identities if I plug my values into the system. In particular, it means that $E_j = b_j$ and $E_k = b_k$ for two specific indexes j and k. I know that I can multiply an equality by the same constant, and hence $\alpha E_k = \alpha b_k$. This also means that $E_j + \alpha E_k = b_j + \alpha b_k$, which proves that the second system has the same solution as the first one. First claim is proved.

($B \Longrightarrow A$) How to go in the opposite direction? The key fact to note is that the operation "multiply by a constant and add to another equation" has a natural *inverse* operation. If I multiply the k-th equation of the second system by $-\alpha$ and add it to the j-th equation I end up exactly with system (1.8). But I already know, from the previous reasoning, that this operation does not produce or loose new solutions. Hence, I finally and proudly state that my two systems of equation have the same solutions. ■

Of course, this is a very simple example of a mathematical proof, but it gives you a general idea of a structure of a typical proof. Almost always we must clearly identify what we need to prove in the form "if A then B," and use the known information (as I used, e.g., the fact that if $a = b$ then $\alpha a = \alpha b$ for any constant α) to get to the conclusion. How detailed a proof is depends on the expected audience. Here I tried to be too detailed, especially with the statement of the above proposition. As the course progresses, my proofs will be more concise and definitely more substantial.

To finish this first introductory lecture I would like to *define* other *elementary row operations.*

Definition 1.4. *Given a system of linear equations, an elementary row operation is one of the following:*

- *Changing the order of the equations in the system.*
- *Multiplying one equation by a non-zero constant.*
- *Adding one equation to another, multiplied by a constant.*

It should be almost immediately clear, after the proof of Proposition 1.2, that the following proposition holds.

Proposition 1.5. *If one linear system can be obtained from another one by a finite sequence of elementary row operations, then they have the same solutions.*

I invite the reader to carefully write arguments that give a *proof* of this proposition.